Determination of Economic Production Quantity with Regard to Machine Failure
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ABSTRACT- In this paper, we study an economic production quantity (EPQ) problem in which an item is produced by a single machine. We assume that the machine may fail in an exponential random time. The repair time is an exponential random variable. It is possible to satisfy the demand from an outside supplier when the machine is failed. The objective is to determine the production and order quantities in each cycle time to minimize the total inventory cost included set-up, ordering, holding, lost sales, production, and purchase costs. Since the cost function is a nonlinear mathematical function we propose an algorithm based on differentiation to obtain the optimum solution.

Keywords— Production, Inventory, Machine failure

I. INTRODUCTION

Economic production quantity (EPQ) is one of the classic models in the inventory literature. In this model there is a machine that produces an item to satisfy the demand. The demand rate and the production rate are known and constant. The objective is to determine the production quantity in each cycle to minimize the inventory system costs included set-up and holding costs.

Many researches were done to integrate the EPQ model with practical applications. Pasandideh et. al. [1] insert imperfect production and rework to the EPQ model. Siajadi et. al. [2] investigate the production cycle of finished item and the joint replenishment cycle of raw materials in an EPQ model. Liao [3] studies an EPQ model by considering maintenance and production programs for an imperfect process involving a deteriorating production system with increasing hazard rate. Teng et. al. [4] propose an EPQ model which is suitable for any given time horizon in any product life cycle including high-tech products. They assume that the demand function and the purchase cost are positive and fluctuating with time. Sarkar and Sarkar [5] develop an EPQ model with deterioration and exponential demand in a production system over a finite time horizon under the effect of inflation and time value of money. They consider the production rate as a dynamic and varying with time variable. Chang et. al. [6] consider an EPQ model for a two-stage assembly system with imperfect processes and variable production rate. They formulate the proposed problem as a cost minimization model where the assembly rate and the production run time of each component process are decision variables. Giri et. al. [7] study the EPQ problem for an unreliable machine in which the failure and repair time are random variable. The production rate is treated as a decision variable. As the stress condition of the machine changes with the production rate, the failure rate of the machine is assumed to be dependent on the production rate.

This paper is an extension on Giri et. al.’s work. In our model, it is possible to satisfy the demand from an outside supplier when the machine is failed. We expect to prevent the lost sales and reduce the total inventory cost with this policy.

The rest of paper is organized as follows. Section 2 is devoted to problem definition. We formulate the problem as a mathematical programming in section 3. In section 4, we develop an algorithm to solve the problem. Section 5 gives the numerical results. Finally, section 6 provides conclusions and suggestions for future research.

II. PROBLEM DEFINITION

We consider the EPQ model in which the machine may fail during the production run. When the machine is failed it is possible to satisfy the demand from an outside supplier. The failure time is an exponential random variable. The repair time is also a random variable with exponential distribution that is independent on the failure time. When a machine failure occurs, the demand is met first from the inventory accumulated during the production run. If the inventory reaches to zero the shortage occurs, in this time the manager decides how much and when an order must be supplied from the outside. It is assumed that at most one order can be placed during the repair time. The objective is to determine production quantity, ordering time and amount of order in each cycle to minimize the total inventory cost included set-up, ordering, holding, lost sales, production, and purchase costs. The model is formulated on the basis of the following assumptions:

1- The planning horizon is infinite.
2- The problem is single item.
3- The item is produced by a single machine.
4- The production rate is greater than the demand rate.
5- The machine set-up time is zero.
6- The lead time (for outside order) is negligible.
7- Unsatisfied demand is lost.

III. MATHEMATICAL MODELLING

Parameters:

P: production rate (item unit/time unit),
\(D\): demand rate (item unit/time unit),
\(A_1\): set-up cost ($/setup),
\(A_2\): ordering cost ($/order)
\(h_1\): holding cost rate of production item ($/item unit/time unit),
\(h_2\): holding cost rate of ordering item ($/item unit/time unit),
\(C_1\): price of production per unit ($/item unit),
\(C_2\): price of order per unit ($/item unit),
\(\pi\): lost sales cost per unit ($/item unit)
\(r\): failure time (exponential random variable with rate \(\alpha\)),
\(s\): repair time (exponential random variable with rate \(\beta\)),

Variables:
\(t_p\): duration of production run (time unit),
\(t_1\): length of time that the machine stops and the demand is met by cumulated production (time unit),
\(t_2\): length of time that the machine stops and the demand is met by received order (time unit),
\(ET\): expected cycle time (time unit),
\(ETIC\): expected total inventory cost per cycle time ($/cycle time),
\(ETICPT\): expected total inventory cost per time unit ($/time unit),

Decision variables:
\(Q_1\): production quantity (item unit),
\(Q_2\): order quantity (item unit),
\(B\): re-order point (item unit),

As the demand rate is constant, we have \(r=B/D\), where \(r\) is duration of cycle time that there is no inventory on hand.

The cycle time is the time interval between two consecutive production start times. The total inventory cost per a cycle time and the cycle time are random variables because the machine failure and the repair time are random variables. Since the planning horizon is infinite, the cost function is expressed as the expected total inventory cost per time unit \((ETICPT)\). The \(ETICPT\) is derived by the expected total inventory cost per a cycle time divided by the expected cycle time.

\[
ETICPT = \frac{E(\text{total inventory cost per a cycle time})}{E(\text{cycle time})} \tag{1}
\]

Depending on the machine failure and the repair time, we will have five cases for the cycle time and the total inventory cost per a cycle time denoted by \(T_i\) and \(TIC_i\), \((i=a,b,c,d,e)\), respectively.

We derive the relations of \(T_i\) and \(TIC_i\) with respect to the parameters and the decision variables in each case.

Case a (Fig. 1): in this case the machine does not fail during the production run, \((t_p>r)\).

\[
T_a = \frac{Q_1}{D} \tag{2}
\]
\[
TIC_a = A_1 + \frac{h_1(P-D)Q_1^2}{2PD} + C_1Q_1 \tag{3}
\]

Case b (Fig. 2): in this case the machine fails during the production run but it repairs before coming down the inventory to zero, \((t_p<s\text{ and } s<t_1)\).

\[
T_b = \frac{Pt}{D} \tag{4}
\]
\[
TIC_b = A_1 + \frac{h_1}{2} \left[ \frac{P^2(P-D)}{D} \right] + C_1Pt \tag{5}
\]

Case c (Fig. 3): in this case machine fails during the production run and the repair time takes so long that the shortage occurs, but the machine repairs before the shortage reaches to \(B\), \((s<t_p\text{ and } t_1<s+t_1+r)\).

\[
T_c = t + s \tag{6}
\]
\[
TIC_c = A_1 + \frac{h_1}{2} \left[ \frac{P^2(P-D)}{D} \right] + \pi[sD - t(P - D)] + C_1Pt \tag{7}
\]
Case d (Fig. 4): in this case machine fails during the production run and the repair time takes so long that shortage reaches to $B$, an order is placed and the machine repairs before the inventory comes down to zero, $(t < t_p$ and $s > t_1 + r + t_2)$.

$$T_d = t + s$$  \hspace{1cm} (8)

$$TIC_d = A_1 + A_2 + \frac{h_1}{2} \frac{t^2}{D} (P-D) + \frac{h_2}{2D} + nD + C_1 P + C_2 Q_2$$  \hspace{1cm} (9)

Case e (Fig. 5): in this case machine fails during the production run and the repair time takes so long that shortage reaches to $B$, an order is placed and the machine repairs after the inventory comes down to zero, $(t < t_p$ and $s > t_1 + r + t_2)$.

$$T_e = t + s$$  \hspace{1cm} (10)

$$TIC_e = A_1 + A_2 + \frac{h_1}{2} \frac{t^2}{D} (P-D) + \frac{h_2}{2D} + nD - t(P-D) - Q_2 + C_1 P + C_2 Q_2$$  \hspace{1cm} (11)

By conditioning on the machine failure time and the repair time, the expected cycle time can be obtained as follows.

$$ET = \int_{t_p}^{t_{1+r}} \left[ \int_{0}^{t_1} (T_b) f(s) ds + \int_{t_1}^{t_{1+r}} (T_e) f(s) ds + \int_{t_1+r}^{t_{1+r+t_2}} (T_d) f(s) ds + \int_{t_{1+r+t_2}}^{\infty} (T_e) f(s) ds \right] f(t) dt + \int_{t_p}^{t_{1+r+t_2}} (T_e) f(t) dt$$  \hspace{1cm} (12)

In Eq. (12), $T_p$, $T_e$, $T_c$, $T_d$, and $T_e$ are substituted by Eq. (2), (4), (6), (8), and (10), respectively. We also substitute $t_p$, $t_1$, and $t_2$ by following relations.

$$t_p = \frac{Q_2}{P}$$  \hspace{1cm} (13)

$$t_1 = \frac{t(P-D)}{D}$$  \hspace{1cm} (14)

$$t_2 = \frac{Q_2}{D}$$  \hspace{1cm} (15)
Finally, we have

$$ET = \frac{P}{D} \left(1 - e^{-\frac{Q_1}{P}}\right) + \frac{U}{v} \left(1 - e^{-\frac{Q_2}{P}}\right)$$

(16)

Where,

$$U = a \left[\frac{1}{\beta} + \frac{Q_2}{D} e^{-\beta r} - \frac{1}{\beta} e^{-\beta r} + \frac{1}{\beta} e^{-\beta (\frac{Q_2}{D} + r)}\right]$$

(17)

$$V = \beta \left(\frac{P - D}{D}\right) + \alpha$$

(18)

By conditioning on the machine failure time and the repair time, the expected total inventory cost per a cycle time can be obtained as follows.

$$ETIC = \int_0^T \left[\int_0^{t_1} (TIC_a)f(s)ds + \int_{t_1}^{t_1 + t_2} (TIC_b)f(s)ds + \int_{t_1 + t_2}^{t_2 + t_3} (TIC_c)f(s)ds\right] f(t)dt + \int_{t_2}^T (TIC_e)f(t)dt$$

(19)

By substituting Eq. (3), (5), (7), (9), (11), (13), (14), and (15) for $TIC_a, TIC_b, TIC_c, TIC_d, TIC_e, tp, t_1$, and $t_2$, respectively, one can derive the $ETIC$ as follows.

$$ETIC = A_1 + \frac{Ph_1}{\alpha^2} + \frac{C_2}{\alpha} - \left(\frac{C_p}{\alpha} + \frac{h_1}{\alpha} (Q_1 + \frac{P}{\alpha})\right) e^{-\frac{Q_1}{P}} + \frac{U}{v} \left(1 - e^{-\frac{Q_2}{P}}\right)$$

(20)

Where,

$$W = a \left[\frac{\pi}{\beta} + \left(A_2 + \frac{h_3 Q_2}{2D}\right) e^{-\beta r} + \left(\frac{\pi D}{\beta}\right) e^{-\beta (\frac{Q_2}{D} + r)}\right]$$

(21)

Finally, the cost function which is expected total inventory cost per time is:

$$ETICPT = \frac{A_1 + \frac{Ph_1}{\alpha^2} + 2\frac{C_2}{\alpha} - \left(\frac{C_p}{\alpha} + \frac{h_1}{\alpha} (Q_1 + \frac{P}{\alpha})\right) e^{-\frac{Q_1}{P}} + \frac{U}{v} \left(1 - e^{-\frac{Q_2}{P}}\right)}{\frac{P}{D} \left(1 - e^{-\frac{Q_1}{P}}\right) + \frac{U}{v} \left(1 - e^{-\frac{Q_2}{P}}\right)}$$

(22)

IV. 3- SOLUTION APPROACH

The $ETICPT$ is a differentiable function. Hence, the necessary conditions for optimum solution are:

$$\frac{\partial ETICPT}{\partial Q_1} = 0$$

(23)

$$\frac{\partial ETICPT}{\partial Q_2} = 0$$

(24)

$$\frac{\partial ETICPT}{\partial r} = 0$$

(25)

There are just two roots, $r=0$ and $r=\infty$ for Eq. (25). $r=\infty$ means that an order is never placed. It indicates that $Q_2=0$. In this case $Q_1$ remains only as a decision variable that can be found from Eq. (23). For $r=0$ the optimum values of $Q_1$ and $Q_2$ are found through solving simultaneous Eq.s (23) and (24). The Hessian matrix of $ETICPT$ is indefinite with respect to $Q_1$ and $Q_2$. Therefore the values obtain from Eq.s (23) and (24 haven’t got the sufficient condition for optimality. Thus, we find the optimum solution via numerical comparison. The $ETICPT$ is calculated for each solution of $Q_1$ and $Q_2$. Each of which minimizes the $ETICPT$ is optimum solution.

In summary, the solution algorithm is as follows.

Step 1. Set $r=\infty$ and $Q_2=0$.

Step 1-1. Find $Q_1$ from Eq. (23).

Step 1-2. Calculate the $ETICPT$ with $Q_1$ obtained in Step 1-1.

Step 2. Set $r=0$.

Step 2-1. Find $Q_1$ and $Q_2$ from Eq.s (23) and (24).

Step 2-2. Calculate the $ETICPT$ with $Q_1$ and $Q_2$ obtained in Step 2-1.

Step 3. The $Q_1$ and $Q_2$ correspond to the minimum of the $ETICPT$ calculated in step 1-2 and 2-2 is the optimum solution.

V. NUMERICAL EXAMPLE

Consider an example with $D=500$ unit/time unit, $P=800$ unit/ time unit, $\alpha=1$ failure/time unit, $\beta=5$ repair/ time unit, $A_1=100$ $$/set-up, A_2=500$ $$/order, h_1=2$$/unit/ time unit, $h_2=4$ $$/unit/ time unit, C_1=20$$/unit, $C_2=40$$/unit, and $\pi=100$$/unit.

The minimum of the $ETICPT$ is 11366 $$/ time unit and the optimum value of $Q_1$ is 2000. If the machine fails and the inventory comes down to zero an order of $Q_2=140$ is placed. Fig. 6 shows the $ETICPT$ with respect to $Q_1$ and $Q_2$.

![Fig. 2 The ETICPT with respect to Q1 and Q2](image-url)
Table IV contains the sensitivity analysis on the machine failure rate. One can see that less machine failure rate leads to less ordering and reduces the \( ETICPT \). These results support the manager to decide whether or not to improve the machine quality. For example, if the cost of reducing the machine failure rate from 1 to 0.5 failure/time unit is less than 487 $/time unit which is difference between 11366 and 10879 $/time unit then expending this cost is justified to improve the machine quality.

Table \( V \)

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( Q_1 )</th>
<th>( Q_2 )</th>
<th>( ETICPT )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>1100</td>
<td>140</td>
<td>10879</td>
</tr>
<tr>
<td>1</td>
<td>2000</td>
<td>140</td>
<td>11366</td>
</tr>
<tr>
<td>1.5</td>
<td>2600</td>
<td>150</td>
<td>11959</td>
</tr>
<tr>
<td>2</td>
<td>3500</td>
<td>150</td>
<td>12654</td>
</tr>
</tbody>
</table>

If \( \alpha \to 0 \) and \( \beta \to \infty \) then the optimum values of \( Q_1 \) and \( Q_2 \) are 365 and 0 item unit, respectively, and the \( ETICPT \) is 10274 $/time unit. It’s the optimum solution of the classic form of the EPQ model. This result is expected because when \( \alpha \to 0 \) and \( \beta \to \infty \) the proposed model approaches to the classic form of the EPQ model.

VI. CONCLUSION

In this paper the EPQ model with the machine failure is considered. The former researches apply a production policy that incurs shortage when the production is insufficient to satisfy the demand. The policy of incurring is not economically appropriate because of the eventual loss of potential demands in the long run. In this paper, to prevent shortage, we propose a policy that the demand may be responded from an outside supplier when the machine is failed. The objective is to minimize the cost function which is a nonlinear mathematical function. The optimum solution can be found by an algorithm based on the differentiation.

Finally, considering the proposed model in multi-item situation can be a good idea for future research.

REFERENCES


Mohammadali Pirayesh is currently an Associate Professor in the Industrial Engineering Department of Ferdowsi University of Mashhad, Mashhad, Iran. He received his B. Sc., M.Sc. & Ph.D. degrees in Industrial Engineering from Sharif University of Technology, Tehran, Iran, in 2000 and 2002 and 2007, respectively. His current research interests include inventory control and supply chain management.